Theory of Music – Jonathan Dimond

Basic Atonal Theory

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Introduction

The concepts outlined in this unit enable the musician to apply new and relevant tools towards the task of analysing the music of Schoenberg, Berg and Webern, and also towards the task of understanding the music of many other composers (even those not intrinsically “12-tone”) from yet another vantage point.

It is also known that, once applied, these concepts are extremely valuable in improving ones skills outside of analysis, into the areas of ear-training and improvisation.

To completely assimilate these concepts into your musicianship, it is recommended that you:

- Listen to recordings of 12-tone / pan-tonal compositions
- Read 12-tone scores and study them with and without the recordings
- Memorize the theoretical concepts
- Apply the theoretical concepts compositionally
- Apply the theoretical concepts aurally (especially by singing)
- Improvise with the concepts with your instrument
- Do further background reading on the composers and the 12-tone technique

Armed with this wholistic experience, you will begin the journey of making sense of 12-tone / atonal music and its language. Only at this point will you be able to discern what is good atonal music, and what is not so good. Just like with any other musical “style”, in atonal music there exists the human element of choice, taste, and also of overall quality. Until we can discern, we will focus on the aforementioned masters…

The Integer Model of Pitch

Our task is to build up the new theoretical foundation upon which atonal music is created. To that end, we must return to the building-blocks of pitch and intervals. The comparatively long history of tonal theory has left its legacy in the naming of intervals and pitches in a way that creates ambiguities and unintended implications when dealing with new music. Take for example, the measuring of four semitones as a major third, or the functional harmonic (and even melodic) implications of an enharmonic equivalence like C# and Db.

In atonal theory, we assign integers (whole numbers) the task of representing pitches. Integers are powerful representatives of pitch because of their ability to be equally spaced (like their chromatic letter-name counterparts) and their ability to be ordered. Furthermore, integers can be subjected to a rich variety of algebraic procedures to which letter names cannot.

Pitch Integer

Any single note can be represented by an integer. In the integer model of pitch, the convention is to let middle C (C4) equal 0 (zero). Every note around middle C is consequently labelled a positive (if higher) or negative (if lower) integer. Liken this labelling system to MIDI note numbers, where each note in the chromatic spectrum receives a unique number.

Ordered Pitch Interval

We can use a positive or negative integer to represent an interval and its direction – ascending (positive) or descending (negative). This is useful for melodic lines. The formulaic definition for ordered pitch interval is:

$$\mathbf{ip\langle x,y\rangle = y-x}$$

Where:

- ip stands for “pitch interval”
- x is the first note (represented as a pitch integer)
- y is the second note (represented as a pitch integer)
- N.B. this formula uses triangular brackets

Unordered Pitch Interval

We can use a positive integer to represent an interval between two notes. This is useful for harmonic situations (e.g. a dyad). The formulaic definition for an unordered pitch interval is:

$$\mathbf{ip(x,y) = |y-x|}$$

Where:

- ip stands for “pitch interval”
- x and y are the two notes (represented as pitch integers)
- the vertical lines | | mean “the absolute value of” – so all negative results are made positive
- N.B. this formula uses regular brackets

Pitch Class

In atonal music theory we often talk about “pitch classes” rather than “pitches”. This seems redundant at first, but just as the letter name C represents all octaves of the note C, two pitches are “pitch-class equivalent” if they are some number of octaves apart. i.e.

$$\mathbf{a=b \text{ if } a=12n+b}$$

or

$$\mathbf{ip(a,b)=12n}$$

Where:

- ip stands for “pitch interval”
- a and b are the two notes (represented as pitch integers)
- n stands for the number of octaves which separate the pitch classes, and if the second formula is satisfied, will be a non-negative integer (zero or higher)
- N.B. this formula uses regular brackets (i.e. it’s an unordered interval)

Modulo

A related mathematical term is "modulo", meaning "in the measure of". Abbreviated as "mod", we can test for pitch-class equivalence using the formula:

$$\mathbf{a=b \text{ if } a=b \bmod_{12}}$$

Mod_{12} arithmetic is similar to what we deal with on a daily basis with clock time. Clock time is also base 12. If we start a four-hour rehearsal at 10am we know that it finishes at 2pm (10+4=14, 14-12=2). 24 hour time only partly avoids these types of calculations (e.g. 2300+3 hrs = 2600, 2600-2400=0200 i.e. 2am). (It's base 24.)

Test understanding of these theories by the following exercises [See Rahn p.20-24]:

1) Write a chromatic scale from C3 to C5 with pitch integer labels.
2) Write a chromatic scale from G3 to G4 with pitch class labels.
3) Write a table of intervals from a perfect unison to a perfect 15^{th} using unordered pitch intervals.
4) Identify ordered and unordered pitch intervals for a series of note pairs.
5) Improvise "dyad melodies" using two pitch classes at a time.
6) Write 8 octaves of pitch integer to pitch-class conversion around register 4 (3 octaves below and 4 octaves above pitch integers 0-11).
7) Practice applying mod.12 to any integer (negative or positive).

Ordered Pitch-Class Interval

We can use a positive integer to represent the smallest number of chromatic steps required to ascend between two pitch classes. The formulaic definition for ordered a pitch-class interval is:

$$\mathbf{i\langle a,b\rangle = b-a \bmod_{12}}$$

Where:

- i stands for "pitch-class interval"
- a is the first note (represented as a pitch integer)
- b is the second note (represented as a pitch integer)
- N.B. this formula uses triangular brackets

Unordered Pitch-Class Interval

We can use a positive integer to represent the smallest number of chromatic steps required to ascend or descend between two pitch classes. Note that because we are dealing with pitch classes with octave equivalence and ignoring order, we are considering the interval and its complement. This means that the result will always be an integer between 1 and 6 inclusive. The formulaic definition for ordered a pitch-class interval is:

$$\mathbf{i(a,b) = \text{the smaller of}}$$
$$\mathbf{b-a \bmod_{12}}$$
$$\mathbf{\text{and}}$$
$$\mathbf{a-b \bmod_{12}}$$

Where:
- i stands for "pitch-class interval"
- a is the first note (represented as a pitch integer)
- b is the second note (represented as a pitch integer)
- N.B. the complementary interval of b-a is a-b, and the addition of these two pitch classes will equal 12
- N.B. this formula uses regular brackets

Pitch-Class Sets

Analysis of a group of pitches as unordered groups of pitch class can be a useful strategy for determining the pitch material featured in a passage of music, harmonically and/or melodically. A Set is simply an unordered group of pitch class, expressed:

{a,b}

This set can yield the PC intervals of i<a,b> or i<b,a>

- N.B. this formula uses curly brackets, representing the set of pitch classes

The "cardinality" of a set simply refers to the number of members (PC) within a set. 0–the null set; 1–monad; 2–dyad; 3– trichord; 4–tetrachord; 5–pentachord; 6–hexachord; 7–septachord, 8–octachord, 9–nonachord; 10–decachord; 11–undecachord; 12–aggregate.

The members of a PC Set can be listed in any order without changing the identity of a set. However, for analysis and easy comparison it is conventional to list the members in "Normal Form".

Normal Form

Normal form exists when the members of a Pitch-Class Set are arranged in ascending order and most "packed to the left". This means that the integers increase from left to right, and furthermore that the largest interval appears between the right-most member and the first of the Set. (i.e. the Set is in "close position.)
If two or more permutations satisfy this criteria, continue by looking for the ordering that has the smallest interval between the 1st and 2nd members, the 1st and 3rd members, etc. Note:

- Sets expressed in Normal Form use curly brackets around the set of pitch classes, e.g. {1,2,4}
- Once the members of a Set have been arranged in ascending order, only *circular permutations* are to be applied in order to determine Normal Form. E.g. the other options in the above example are {2,4,1} and {4,1,2}.
- To find the total number of arrangements of members in a Set (not just the circular permutations) the factorial equation applies:

$$n! = 1 \times 2 \times \ldots \times n$$

Where:
- *n* is the number of members in the Set.

This formula gives the number of *general permutations*, which of course is a larger number than for circular permutations.

A simple procedure for finding Normal Form is as follows:

1) Arrange the set in increasing numerical order, eliminating any PC duplicates.
2) Identify the (ordered PC) intervals between each member of the set. This includes the interval between the right-most and first member of the set, measured also as an ordered PC interval. (Notice that the sum of all the measured intervals should total 12, being the complete octave.)
3) Place the largest interval on the outside, whilst keeping the PC in ascending order. Rotate the members so that the right-hand member of the pair of notes with the largest interval is first. If there is more than one option, choose the one with the smallest (lowest) initial member.
4) Check whether the smallest interval is first, i.e., whether it is "packed to the left".
   a) If it is not, and there are other rotations that keep an equally large interval on the outside, rotate so that it is first.
   b) If it is last, and no other rotation is available, invert the set by either:
   - rewriting the same series of ordered PC intervals from right to left, starting on the same initial PC. Or;
   - subtracting each PC from 12, rearranging in ascending order, and then rotating to place the largest interval on the outside (as per step 3).

   Of course, the first and last members of the set will remain the same after inversion, as will the outside interval.

Prime Form

A Pitch-Class Set in Normal Form but transposed so that 0 (zero) is its first member is said to be in Prime Form.
Sets expressed in Prime Form use square brackets around the set of pitch classes. E.g. Our Normal Form example {1,2,4} would be [0,1,3] in Prime Form. Prime form sets are really the most useful kind, and therefore the usual goal of analysis.
Exercise: Analyse the trichords from "Nacht" (Pierrot Lunaire) and express them in Prime Form.

Transpositional Type Sets

Once a Set has been expressed in Prime Form, it is used to represent all of the remaining 11 transpositions of the Set. We identify such Sets with the suffix T_n. E.g.

$$\{a,b,c\}\ T_n$$

Where:

- a,b,c represent the PC members
- T stands for Transpositional
- n stands for the number of semitones the set is transposed to, from 0-11.

For example $\{0,4,7\}T_n$ represents the major triad in all possible transpositions.
$\{0,4,7\}T_1$ represents the major triad in its transposition to C# (up 1 semitone).
T_n sets are the most common type of set, and a set written thus [0,1,5] is implied to represent all transpositions of that Prime Form set.

Transpositional/Inversional Type Sets

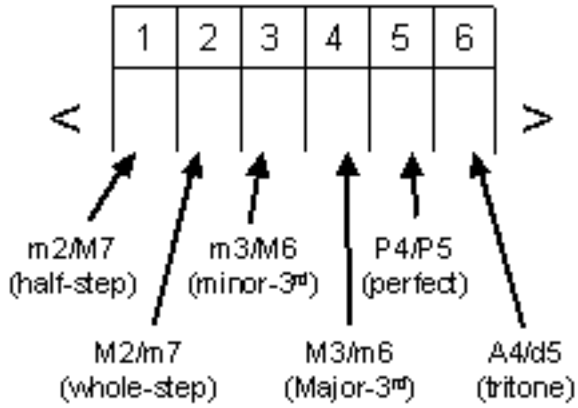
More important than nominating a Prime Form set as a representative of all transpositions of a Set is the ability to nominate a Prime Form set as the representative of all transpositions and inversions of that Set.

Though T_n/T_nI is the usual subscript to indicate such Transpositional/Inversional type sets, in practice a Set expressed in Prime Form with square brackets is typically meant to represent all transpositions and inversions of that Set. (It is said to thus "exhaust the domain").
I.e. [a,b,c] = (a,b,c) T_n/T_nI
This "representative form" is the "most normal form" for any group of pitch class, and all possibilities have been calculated by theorists like Allen Forte and others.

Give out Rahn's table of Transpositional/Inversional Set Types, based on Forte.

This table is organized based upon cardinality, or the number of members of the set, from trichords (three-note chords) to dodecachords (12-note "aggregates"). Notice how the table is organized with the complements of one cardinality of set being written in the adjacent columns (e.g. pentachords and septachords).

Notice also the interval vector column, which defines the intervallic potential of any set by listing the possible unordered PC intervals (interval classes) it can generate.



Exercise.
Create a table with 8 columns and 11 rows.
Use the headings of the columns below and fill in the details in the outside columns.
Complete the remaining columns as per the column headers and refer to the table on that follows to verify your calculations.

ip<x,y>	ip(x,y)	Int.↑	i<a,b>	i(a,b)	Int.↓	ip(x,y)	ip<x,y>
+1							-11
+2							-10
+3							-9
+4							-8
+5							-7
+6							-6
+7							-5
+8							-4
+9							-3
+10							-2
+11							-1

ip<x,y>	ip(x,y)	Int.↑	i<a,b>	i(a,b)	Int.↓	ip(x,y)	ip<x,y>
+1	1	Min 2	1	1	Maj 7	11	-11
+2	2	Maj 2	2	2	Min 7	10	-10
+3	3	Min 3	3	3	Maj 6	9	-9
+4	4	Maj 3	4	4	Min 6	8	-8
+5	5	P4	5	5	P5	7	-7
+6	6	TT	6	6	TT	6	-6
+7	7	P5	7	5	P4	5	-5
+8	8	Min 6	8	4	Maj 3	4	-4
+9	9	Maj 6	9	3	Min 3	3	-3
+10	10	Min 7	10	2	Maj 2	2	-2
+11	11	Maj 7	11	1	Min 2	1	-1

Basic Operations

These are variation techniques that "operate" on a given series of pitches to create another related series. They are techniques for development/variation. Please note that we are talking about a melodic line that has an essential contour, that is, a series of ordered pitch intervals in specific directions from a starting note. We are operating on pitch integers, not a group of unordered pitch class, like a set.

Original

"O" stands for the original series of pitch integers. (You may also see "P" used, meaning "principal"). The original series can be any number of notes in length, from two upwards (and not necessarily a "12-tone row"). You can measure ip<x,y> between each pair of notes in the series to identify the character of the series.

O_n (or P_n) represents the Original Series, starting on PC *n*

Pitch Transposition

Transposition adds an identical number of semitones to every member of a series of pitch integers to uniformly transpose it to another place. The ordered pitch intervals will remain unchanged.

$$T^p_n(x) = x + n$$

Where:

- x represents the pitch (represented as a pitch integer)
- T stands for Transposition
- n stands for the number of semitones the pitch is transposed, and can be any positive or negative integer (representing any interval)
- The formula can be read "T sub *n* maps pitch *x* into pitch *x* + *n*"

An ordered series of pitch intervals can be quickly transposed by transposing the first pitch and then adding the original series of intervals to create the transposed series of pitch intervals.

$T^p_n(x,y,z)$ represents the Original Series, transposed up by *n* semitones

Exercise: Complete exercises from Rahn p.40-42 (Ex.1, 2, 3, and 3-1)

Inversion

When we invert a melodic line we turn the intervals upside-down, i.e. positive intervals become negative and *vice versa*. The intervals are said to be in reverse polarity. It is conventional to start on the same note as the original line, so to be clear in Atonal Theory we talk about "transposed inversion."

$$\mathbf{T^p{}_n\, I\, (x) = {}^{-}x + n}$$

Where:

- x represents the pitch (represented as a pitch integer)
- T stands for Transposition
- I stands for Inversion
- n stands for the number of semitones the pitch is transposed, and can be any positive or negative integer (representing any interval)
- The formula can be read "Pitch Inversion maps pitch *x* into pitch negative *x* + *n*"

An ordered series of pitch integers can be quickly inverted by starting on the first pitch and then adding the opposite polarity of the original series of ordered pitch intervals to create the inverted series of notes. This method of course starts on the same pitch as the original, and so is a transposed inversion. The true "horizon" inferred by the Inversion formula is zero (middle C), because the inverted pitch integer will "flip" to the same distance the opposite direction from this horizon when the formula is applied.

I_n represents the Original Series Inverted, starting on PC *n*

Retrograde

A series of pitch integers played backwards results in a series of ordered pitch intervals that are in reverse order and opposite polarity.

R_n represents the Original Series in Retrograde, ending on PC *n*

Retrograde Inversion

A series of pitch integers with their ordered pitch intervals inverted and then played backwards results in a series of intervals that are in reverse order. In other words, it is the retrograde of the inversion, which means the intervals are in reverse order and reverse of reverse polarity – which cancels out a polarity change.

RI_n represents the Original Series in Retrograde Inversion, ending on PC *n*

Exercise: Apply the Basic Operations to 11 given melody extracts.

Pitch-Class Transposition

So far we've been examining operations on a given series of ordered pitch intervals. Pitch-class transposition adds an identical number of semitones to every member of a Set, but does not preserve the original line's contour because all PC are mod_{12}. Pitch-class transposition does preserve the original line's ordered and unordered PC intervals.

$$\mathbf{T_n(x) = x + n\ (mod_{12})}$$

Where:

- x represents the pitch
- T stands for Transposition
- n stands for the number of semitones the pitch is transposed, and can be any positive or negative integer (representing any interval)

Exercise: Recreate the examples from Rahn p.42 (Ex.4)

Trichords

For ease of application of these theories, both aurally and compositionally, we will focus our attention on three-note chords.
I find it useful to categorize the twelve trichords into three "families" based upon interval content:
Family 1 – contains one 1 (semitone)
Family 2 – contains one 2 (tone) and no 1's
Family 3 – contains neither a 1 (semitone) nor a 2 (tone)
In the following table, I have labelled the trichords in each family with tonal descriptions also, alongside the inversion of the representative form.

FAMILY 1	***T_n/T_nI***	***Inversion***	***Tonal Implication***
3-1	[0,1,2]	(0,1,2)	Chromatic scale
3-2	[0,1,3]	(0,2,3)	Octatonic/Phrygian/Minor
3-3	[0,1,4]	(0,3,4)	Harm. Minor/Bluesy #9
3-4	[0,1,5]	(0,4,5)	Maj.7
3-5	[0,1,6]	(0,5,6)	Leading,Tonic,SubDom./ Dom.,P.4,#4
FAMILY 2			
3-6	[0,2,4]	(0,2,4)	Whole tone scale
3-7	[0,2,5]	(0,3,5)	Maj. & Min. Pentatonic
3-8	[0,2,6]	(0,4,6)	Dom.7/Maj. b5
3-9	[0,2,7]	(0,5,7)	Quartal- Sus 4 root & 1st inv.
FAMILY 3			
3-10	[0,3,6]	(0,3,6)	Diminished triad
3-11	[0,3,7]	(0,4,7)	Minor & Major triad
3-12	[0,4,8]	(0,4,8)	Augmented triad

As the most general and boiled-down form of pitch combinations, these Transpositional/Inversional type sets represent all possibilities of transposition and inversion. It prompts the question: “How many trichords are there?” How many trichords do these 12 Sets actually represent?

Exercise: Calculate the number of trichords – including transpositions and inversions.

How to work this task out:
Of the 12 trichords there are 4 symmetrical and 8 asymmetrical trichords. As can be seen in the table above, there are no inversions of the symmetrical trichords. Trichord 3-9 [0,2,7] is inversionally symmetrical also, because (0,5,7) is simply a rotation of (0,2,7). The inversion creates the same pitch classes (CDG = GCD). Therefore we need to treat it like the 4 symmetrical trichords.
[O,4,8] is also a special case because it is not only symmetrical but divides the octave into equal parts, limiting transposition. It can only be transposed 4 times (because CEG#=EG#C=G#CE).
However, all the other trichords can be transposed to 12 different “keys”.
Any combination of 3 notes can also be reordered 6 times (Using the aforementioned factorial equation *3! = 1x2x3*=6).

So, for the symmetrical trichords [0,1,2] [0,2,4] [0,2,7] [0,3,6] we have:
4 Set types
6 Orderings each
12 Transpositions
4x6x12 = 288 Trichords

And for the symmetrical trichord [O,4,8] we have:
1 Set type
6 Orderings
4 Transpositions
1x6x4 = 24 Trichords

And for the asymmetrical trichords [0,1,3] [0,1,4] [0,1,5] [0,1,6] [0,2,5] [0,2,6] [0,3,7] we have:
7 Set types
6 Orderings each
2 Inversions each
12 Transpositions
7x6x2x12 = 1008 Trichords
Grand Total = 1320 Trichords

Homework Task: For the 11 given melody extracts, work out the pitch sequence in integers, Normal Form, Set Type (Prime Form), Ordered Pitch Intervals, Unordered Pitch Intervals, Ordered Pitch-class Intervals, and Unordered Pitch-class Intervals**.** Optional: If there are 4 or more notes, work out the consecutive trichords.

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